Stationary Palm’s theorem (ref Tijms, website Scribd)

A - Definition of Poisson process for constant $\lambda$:

$$p(\lambda, t, j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t} = \frac{(\lambda t)^j}{j!} \left(1 - \frac{\lambda t}{1!} + \frac{(-\lambda t)^2}{2!} + \frac{(-\lambda t)^3}{3!} + \cdots \right) \quad j = 0, 1, 2, 3, \ldots$$

B - Properties of Poisson process (see numerical example on table below):

For $\Delta t$ (or $\lambda \Delta t$) small, that is $\Delta t \to 0$ and the probability of $j = 0$ (zero) arrival in time $\Delta t$ is

$$p(\lambda, \Delta t, 0) = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + \frac{(-\lambda \Delta t)^2}{2!} + \frac{(-\lambda \Delta t)^3}{3!} + \cdots \Rightarrow p(\lambda, \Delta t, 0) = 1 - \lambda \Delta t + O(\lambda, \Delta t, 0)$$

where

$$O(\lambda, t, 0) = \frac{(-\lambda \Delta t)^2}{2!} + \frac{(-\lambda \Delta t)^3}{3!} + \cdots \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{O(\lambda, \Delta t, 0)}{\Delta t} = 0$$

For $\Delta t$ (or $\lambda \Delta t$) small, that is $\Delta t \to 0$ and the probability of $j = 1$ (one) arrival in time $\Delta t$ is

$$p(\lambda, \Delta t, 1) = \lambda \Delta t e^{-\lambda \Delta t} = \lambda \Delta t (1 - \lambda \Delta t + \cdots) \Rightarrow p(\lambda, \Delta t, 1) = \lambda \Delta t + O(\lambda, \Delta t, 1)$$

For $\Delta t$ (or $\lambda \Delta t$) small, that is $\Delta t \to 0$ and the probability of $j = 2$ (two) arrivals in time $\Delta t$ is

$$p(\lambda, \Delta t, 2) = \frac{(\lambda \Delta t)^2}{2!} e^{-\lambda \Delta t} = \frac{(\lambda \Delta t)^2}{2!} (1 - \lambda \Delta t + \cdots) \Rightarrow p(\lambda, \Delta t, 2) = O(\lambda, \Delta t, 2)$$

For $\Delta t$ (or $\lambda \Delta t$) small, that is $\Delta t \to 0$ and the probabilities of $j = 3, 4, \ldots$ arrivals in time $\Delta t$ as similar to $p_2(\Delta t)$ are

$$p(\lambda, \Delta t, j) = O(\lambda, \Delta t, j) \quad \text{for} \; j = 3, 4, 5, \ldots$$

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<table>
<thead>
<tr>
<th>j</th>
<th>( \lambda )</th>
<th>( \Delta t ) (small)</th>
<th>( \lambda \Delta t )</th>
<th>( \text{poisson}(j, \lambda \Delta t) )</th>
<th>Approximation</th>
<th>( \text{poisson}(j, \lambda \Delta t) / \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.000200</td>
<td>1 – ( \lambda \Delta t = 1 – 0.002 )</td>
<td>0.000200</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>( \lambda \Delta t = 0.002 )</td>
<td>0.0006</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>5</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>6</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
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<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>8</td>
<td>2.0</td>
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<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>9</td>
<td>2.0</td>
<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
<tr>
<td>10</td>
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<td>0.001</td>
<td>0.002</td>
<td>0.0006</td>
<td>Very small</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

C – Time dependent probability equation

![Diagram of time dependent probability equation](https://via.placeholder.com/150)

<table>
<thead>
<tr>
<th>time t</th>
<th>time t</th>
<th>interval ( \Delta t )</th>
<th>interval ( \Delta t )</th>
<th>interval ( \Delta t )</th>
<th>time ( (t + \Delta t) )</th>
<th>time ( (t + \Delta t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_j(t) )</td>
<td>0 arrival</td>
<td>( p(\lambda, \Delta t, 0) )</td>
<td>( p_j(t + \Delta t) )</td>
<td>( p_j(t + \Delta t) )</td>
<td>( p_j(t + \Delta t) )</td>
<td>( G(t + \Delta t) )</td>
</tr>
<tr>
<td>( p_j(t) )</td>
<td>( G(t) )</td>
<td>1 arrival</td>
<td>( p(\lambda, \Delta t, 1) )</td>
<td>1 service</td>
<td>( p_j(t + \Delta t) )</td>
<td>( G(t + \Delta t) )</td>
</tr>
<tr>
<td>( p_{j-1}(t) )</td>
<td>( G(t) )</td>
<td>1 arrival</td>
<td>( p(\lambda, \Delta t, 1) )</td>
<td>0 service</td>
<td>( p_j(t + \Delta t) )</td>
<td>( G(t + \Delta t) )</td>
</tr>
<tr>
<td>2 arrivals</td>
<td>( p(\lambda, \Delta t, 2) = O(\lambda, \Delta t, 2) )</td>
<td>( O(\lambda, \Delta t, 2) )</td>
<td>( G(t + \Delta t) )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 arrivals</td>
<td>( p(\lambda, \Delta t, 3) = O(\lambda, \Delta t, 3) )</td>
<td>( O(\lambda, \Delta t, 3) )</td>
<td>( G(t + \Delta t) )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( JFukuda \ Jan2009 \)
$p_j(t)$, $p_{j-1}(t), p_j(t + \Delta t)$ are probabilities (service time process) of $j, j-1, j$ servers busy respectively at times $t, t, t + \Delta t$

$G(t), G(t + \Delta t)$ are any service time probability distribution respectively at times $t, t + \Delta t$

$[1 - G(t)], [1 - G(t + \Delta t)]$ are survival time probability distribution respectively at times $t, t + \Delta t$

$p(\lambda, \Delta t, 0), p(\lambda, \Delta t, 1), p(\lambda, \Delta t, 2), p(\lambda, \Delta t, 3)$ are probabilities of $0, 1, 2, 3$ arrivals (Poisson process) in interval $\Delta t$

$O(\lambda, \Delta t, 2), O(\lambda, \Delta t, 3)$ are very small and negligible (Poisson process) values

Taking into account the above mutually exclusive cases and summarizing results from (B) above

0 (zero) customer arrival in interval $\Delta t$

$p(\lambda, \Delta t, 0) = 1 - \lambda \Delta t + O(\lambda, \Delta t, 0)$

$P(p(\lambda, \Delta t, 0), p_j(t)) = [1 - \lambda \Delta t]p_j(t) + O(\lambda, \Delta t, 0)p_j(t)$

1 customer arrival in interval $\Delta t$ and 1 service concluded in $\Delta t$

$p(\lambda, \Delta t, 1) = \lambda \Delta t + O(\lambda, \Delta t, 1)$

$P(p(\lambda, \Delta t, 1), p_j(t), 1 \text{ service concluded}) = \lambda \Delta t p_j(t) G(t + \Delta t) + O(\lambda, \Delta t, 1)p_j(t)$

1 customer arrival in interval $\Delta t$ and 0(zero) service concluded in $\Delta t$

$p(\lambda, \Delta t, 1) = \lambda \Delta t + O(\lambda, \Delta t, 1)$

$P(p(\lambda, \Delta t, 1), p_{j-1}(t), 0 \text{ service concluded}) = \lambda \Delta t p_{j-1}(t)[1 - G(t + \Delta t)] + O(\lambda, \Delta t, 1)p_{j-1}(t)$
Summing the cases

\[ p_j(t + \Delta t) = P(p(\lambda, \Delta t, 0), p_j(t)) + P(p(\lambda, \Delta t, 1), p_j(t), 1 \text{ service concluded }) + \]
\[ + P(p(\lambda, \Delta t, 1), p_{j-1}(t), 0 \text{ service concluded }) + O(\lambda, \Delta t) \]

\[ p_j(t + \Delta t) = (1 - \lambda \Delta t)p_j(t) + \lambda \Delta t G(t + \Delta t)p_j(t) + \lambda \Delta t (1 - G(t + \Delta t))p_{j-1}(t) + O(\lambda, \Delta t) \]

**D – Arrangement of differential equations system**

From (C) above

\[ p_j(t + \Delta t) = (1 - \lambda \Delta t)p_j(t) + \lambda \Delta t G(t + \Delta t)p_j(t) + \lambda \Delta t (1 - G(t + \Delta t))p_{j-1}(t) + O(\lambda, \Delta t) \]

Subtracting \( p_j(t) \) from \( p_j(t + \Delta t) \), dividing by \( \Delta t \) and doing \( \Delta t \to 0 \)

\[ \frac{p_j(t + \Delta t) - p_j(t)}{\Delta t} = -\lambda p_j(t) + \lambda G(t + \Delta t)p_j(t) + \lambda (1 - G(t + \Delta t))p_{j-1}(t) + \frac{O(\lambda, \Delta t)}{\Delta t} \]

\[ \lim_{\Delta t \to 0} \frac{p_j(t + \Delta t) - p_j(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[ -\lambda p_j(t) + \lambda G(t + \Delta t)p_j(t) + \lambda (1 - G(t + \Delta t))p_{j-1}(t) + \frac{O(\lambda, \Delta t)}{\Delta t} \right] \]

\[ \lim_{\Delta t \to 0} \frac{p_j(t + \Delta t) - p_j(t)}{\Delta t} = p_j'(t) = -[\lambda (1 - G(t))]p_j(t) + \lambda [1 - G(t)]p_{j-1}(t) \]
\begin{align*}
\eta(t) &= \lambda [1 - G(t)] \\
p_0'(t) &= -\eta(t)p_0(t) \\
p_1'(t) &= -\eta(t)p_1(t) + \eta(t)p_0(t) \\
p_2'(t) &= -\eta(t)p_2(t) + \eta(t)p_1(t) \\
p_3'(t) &= -\eta(t)p_3(t) + \eta(t)p_2(t) \\
\ldots \ldots \\
p_j'(t) &= -\eta(t)p_j(t) + \eta(t)p_{j-1}(t)
\end{align*}

E – Solution of differential equations system

\begin{align*}
\Lambda(t) &= \int_0^t \eta(t) dt = \lambda \int_0^t [1 - G(t)] dt \implies \Lambda'(t) = \eta(t) \\
f(x) &= e^{g(x)} \implies f'(x) = g'(x)e^{g(x)} = g'(x)f(x)
\end{align*}

For \( j = 0 \)
\begin{align*}
p_0'(t) &= -\eta(t)p_0(t) = -\Lambda'(t)p_0(t) \implies p_0(t) = e^{-\Lambda(t)}
\end{align*}

For \( j = 1 \)
\begin{align*}
p_1'(t) &= -\eta(t)p_1(t) + \eta(t)p_0(t) = \Lambda'(t)p_0(t) - \Lambda'(t)p_1(t) = \Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)p_1(t)
\end{align*}

\begin{align*}
[\alpha\beta]' &= \alpha'\beta + \alpha\beta' \\
\alpha' &= \Lambda'(t) \implies \alpha = \Lambda(t) \text{ and } \beta = e^{-\Lambda(t)}
\end{align*}
\[ [\alpha \beta]' = \left[ \Lambda(t)e^{-\Lambda(t)} \right]' = \Lambda'(t)e^{-\Lambda(t)} + \Lambda(t)(-\Lambda'(t))e^{-\Lambda(t)} \Rightarrow p_1(t) = \Lambda(t)e^{-\Lambda(t)} \]

For \( j = 2 \)

\[ p_2'(t) = -\eta(t)p_2(t) + \eta(t)p_1(t) = \Lambda'(t)p_1(t) - \Lambda(t)p_2(t) = \Lambda'(t)\Lambda(t)e^{-\Lambda(t)} - \Lambda'(t)p_2(t) \]

\[ [\alpha \beta]' = \alpha'\beta + \alpha\beta' \quad \alpha' = \Lambda'(t) \Rightarrow \alpha = \Lambda(t) \text{ and } \beta = \Lambda(t)e^{-\Lambda(t)} \]

\[ [\alpha \beta]' = \left[ \Lambda^2(t)e^{-\Lambda(t)} \right]' = 2\Lambda(t)\Lambda'(t)e^{-\Lambda(t)} + \Lambda^2(t)(-\Lambda'(t))e^{-\Lambda(t)} \]

\[ \left[ \frac{\alpha \beta}{2} \right]' = \Lambda(t)\Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)\left[ \frac{\Lambda^2(t)e^{-\Lambda(t)}}{2} \right] \Rightarrow p_2(t) = \frac{\Lambda^2(t)}{2}e^{-\Lambda(t)} \]

For \( j = 3 \)

\[ p_3'(t) = -\eta(t)p_3(t) + \eta(t)p_2(t) = \Lambda'(t)p_2(t) - \Lambda(t)p_3(t) = \Lambda'(t)\left( \frac{\Lambda^2(t)}{2}e^{-\Lambda(t)} - \Lambda(t)p_2(t) \right) \]

\[ [\alpha \beta]' = \alpha'\beta + \alpha\beta' \quad \alpha' = \Lambda'(t) \Rightarrow \alpha = \Lambda(t) \text{ and } \beta = \frac{\Lambda^2(t)}{2}e^{-\Lambda(t)} \]

\[ [\alpha \beta]' = \left[ \frac{\Lambda^3(t)}{2}e^{-\Lambda(t)} \right]' = 3\frac{\Lambda^2(t)}{2}\Lambda'(t)e^{-\Lambda(t)} + \frac{\Lambda^3(t)}{2}(-\Lambda'(t))e^{-\Lambda(t)} \]

\[ \left[ \frac{\alpha \beta}{3} \right]' = \frac{\Lambda^2(t)}{2}\Lambda'(t)e^{-\Lambda(t)} - \Lambda'(t)\left[ \frac{\Lambda^3(t)e^{-\Lambda(t)}}{2 \cdot 3} \right] \Rightarrow p_3(t) = \frac{\Lambda^3(t)}{2 \cdot 3}e^{-\Lambda(t)} \]
For \( j \), by induction

\[
p_j(t) = \frac{\lambda^j(t)}{j!} e^{-\lambda(t)}
\]

\[F - \text{Conclusion}\]

Under condition of stationary and steady states \((t \to \infty)\) the service probabilities are

\[
\Lambda(t \to \infty) = \lambda \left[ \lim_{t \to \infty} \int_0^t [1 - G(t)]dt \right] = \lambda \int_0^\infty [1 - G(t)]dt
\]

By definition \( \int_0^\infty [1 - G(t)]dt = E[G(t)] = \mu = \text{expeced or mean service time} \)

\[
p(\lambda, \mu, j) = \frac{(\lambda \mu)^j}{j!} e^{-\lambda \mu} \quad \text{for } j = 0, 1, 2, 3, \ldots
\]

\[G - \text{Completion}\]

The dynamic and not stationary conditions of theorem (without demonstration) are

\[
p(j, t) = \frac{[\Lambda(t)]^j}{j!} e^{-\Lambda(t)}
\]

\[
\lambda = \lambda(\tau)
\]

\[
\eta(\tau, t) = \lambda(\tau)[1 - G(t - \tau)]
\]
\[ \Lambda(t) = \int_0^t \eta(\tau, t)\,d\tau = \int_0^t \lambda(\tau)[1 - G(t - \tau)]\,d\tau \]

\[ p(j, t) = \frac{\left[\int_0^t \lambda(\tau)[1 - G(t - \tau)]\,d\tau\right]^j}{j!} e^{-\int_0^t \lambda(\tau)[1 - G(t - \tau)]\,d\tau} \]

Doing \( \lambda(t) = \lambda = \text{constant} \)

\[ \theta = t - \tau \Rightarrow d\theta = -d\tau, \quad \tau = 0 \Rightarrow \theta = t, \quad \tau = t \Rightarrow \theta = 0 \quad \text{and} \quad t \to \infty \quad \text{(steady state)} \]

\[ \lim_{t \to \infty} \left[\int_0^t \lambda(\tau)[1 - G(t - \tau)]\,d\tau\right] = \lim_{t \to \infty} \left[\lambda \int_0^t [1 - G(\theta)](-d\theta)\right] = \lambda \lim_{t \to \infty} \left[\int_0^t [1 - G(\theta)]\,d\theta\right] = \lambda \int_0^\infty [1 - G(\theta)]\,d\theta \]

And the result is the same as got in (F) above.