Stationary Palm’s theorem

The result of this theorem is fundamental of supply and repair theory. Its demonstration herein is based on Barlow and Proschan, in their book of 1975 they did not call it Palm’s theorem.

\[ p(\lambda, x_1 - 0) p(\lambda, x_2 - x_1) \ldots p(\lambda, x_k - x_{k-1}) p(\lambda, t_0 - x_k) = \]

\[ = \lambda^k \exp(-\lambda t_0) / k! \cdot \exp(-\lambda_t) \]

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\[ \lambda \text{ events of Poisson process during } [0, t_0] \text{ with constant mean rate } \lambda. \]

\[ x_1 - 0, x_2-x_1, \ldots, x_k-x_{k-1}, t_0-x_k \text{ successive intervals of occurrence with } 0 \leq x_1 \leq x_2 \leq \ldots \leq x_{k-2} \leq x_{k-1} \leq x_k \leq t_0. \]

Poisson process \(\Rightarrow\) by definition inter-event times are exponentially distributed with parameter \(\lambda\).

The conditional joint density is:

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<table>
<thead>
<tr>
<th>time</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>arrivals</td>
<td>( i + n )</td>
</tr>
</tbody>
</table>

\[
\text{prob } (\lambda t; n + i) = \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t}
\]

By \( n \) Bernoulli trials for iid rv:

\[
\binom{n+i}{i} (P(t))^i (1 - P(t))^{n-i}
\]

If failures occur according to a stationary Poisson process and repair times are independent, identically distributed random variables with a finite mean (MTTR), the steady-state probability distribution of the numbers of units in repair is Poisson and is independent of the particular service distribution.
\[ p(\lambda t, P(t); n) = \sum_{i=0}^{\infty} \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t} \left( \frac{n+i}{n} \right) (P(t))^{n} (1 - P(t))^i = \sum_{i=0}^{\infty} \frac{(\lambda t)^{n+i}}{(n+i)!} e^{-\lambda t} \frac{(n+i)!}{n!} (P(t))^n (1 - P(t))^i = \]

\[ = \frac{(\lambda t)^n}{n!} e^{-\lambda t} (P(t))^n \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} (1 - P(t))^i = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} (1 - P(t))^i = \]

\[ = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{[\lambda t(1 - P(t))]^i}{i!} \]

\[ e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]  

if \[ \sum_{i=0}^{\infty} \frac{[\lambda t(1 - P(t))]^i}{i!} = e^{\lambda t(1 - P(t))} \]

\[ p(\lambda t, P(t); n) = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{[\lambda t(1 - P(t))]^i}{i!} = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda t} e^{\lambda t(1 - P(t))} = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda tP(t)} \]

From the table below

\[ \lim_{t \to \infty} p(\lambda t, P(t); n) = \lim_{t \to \infty} \frac{(\lambda tP(t))^n}{n!} e^{-\lambda tP(t)} = \frac{(\lambda tP(t))^n}{n!} e^{-\lambda tP(t)} = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \Rightarrow p(\lambda, \nu; n) = \frac{\lambda^n}{n!} e^{-\lambda t} \]

\[ p(\lambda, \nu; n) = \frac{\lambda^n}{n!} e^{-\lambda t} \]
<table>
<thead>
<tr>
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<th>General distribution</th>
<th>Exponential distribution</th>
</tr>
</thead>
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<tr>
<td><strong>Life distribution</strong></td>
<td>$G(t) = 1 - G(t)$</td>
<td>$1 - e^{-\frac{t}{\nu}}$</td>
</tr>
<tr>
<td><strong>Survival probability</strong></td>
<td>By definition: $\int_{0}^{\infty} G(t) dt = \int_{0}^{\infty} (1 - G(t)) dt = E(t) = \nu$</td>
<td>$e^{-\frac{t}{\nu}}$</td>
</tr>
<tr>
<td></td>
<td>$\nu = \text{constant} = \text{mean repair time}$</td>
<td>$\lim_{t \to \infty} \int_{0}^{t} e^{-\frac{t}{\nu}} dt = \lim_{t \to \infty} \left[ -\nu \left( e^{-\frac{t}{\nu}} - 1 \right) \right] = \nu$</td>
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<td><strong>Density probability</strong></td>
<td>$g(t) = \frac{d}{dt} \left[ G(t) \right]$</td>
<td>$\left( \frac{1}{\nu} \right) e^{-\frac{t}{\nu}}$</td>
</tr>
<tr>
<td></td>
<td>$p(t) = \frac{G(t-x)}{t}$</td>
<td>$\left( \frac{1}{t} \right) e^{-\frac{(t-x)}{\nu}}$</td>
</tr>
<tr>
<td><strong>$P(t) = \int_{0}^{t} p(t) dt$</strong></td>
<td>$t-x = u \Rightarrow -dx = du$</td>
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<td><strong>Time average value of $\tilde{G}(t) = 1 - G(t)$</strong></td>
<td>$\int_{0}^{t} \frac{\tilde{G}(t-x)}{t} dx = \int_{t}^{0} \tilde{G}(u) (- du)$ = 0</td>
<td>$\tilde{e}^{-\frac{(t-x)}{\nu}} = \int_{t}^{0} \tilde{e}^{-\frac{u}{\nu}} du = -\frac{V}{t} \left[ e^{-\frac{u}{\nu}} \right]_{0}^{t}$</td>
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<td>$= \frac{1}{t} \int_{0}^{t} \tilde{G}(u) du = \frac{1}{t} \int_{0}^{t} \tilde{G}(u) du$</td>
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<td><strong>Under steady state</strong></td>
<td>$\lim_{t \to \infty} \int_{0}^{t} \tilde{G}(u) du = \nu$</td>
<td>$\lim_{t \to \infty} \left[ \nu \left( 1 - e^{-\frac{t}{\nu}} \right) \right] = \nu$</td>
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**Note:** The Palm’s theorem is proved for any distribution $\tilde{G}(t)$ with mean $\nu$ (repair time, MTTR), the exponential time distribution is shown only for comparison purpose.